ELEN 4810 Midterm Exam

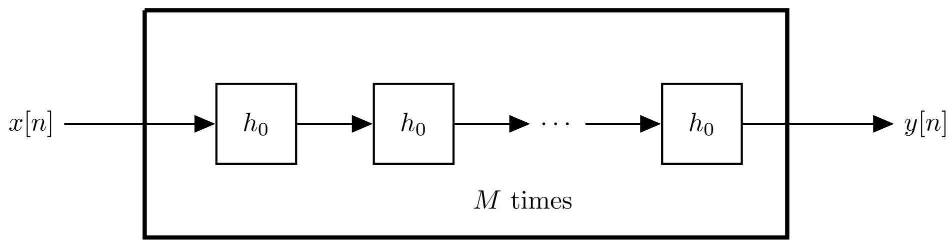
Monday, October 28, 2024, 4:10-6:10 PM. One sheet of handwritten notes is allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 3 questions. Good luck!

Name:

Uni:

1. Systems in Time and Frequency. Consider the following linear, time invariant system:



with

$$h_0[n] = \begin{cases} -1 & n = 0 \\ \sqrt{2} & n = 1 \\ -1 & n = 2 \\ 0 & \text{else} \end{cases}$$

Please answer the following questions:

Part (a). Is this system **causal**? Why or why not?

Part (b). Is this system **stable**? Why or why not?

Part (c). What is the **frequency response** $H(e^{j\omega})$ of the overall system?

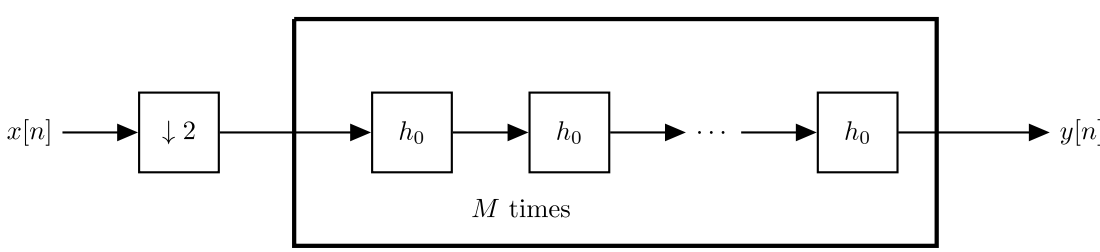
Part (d). What is the **output** $y[n]$ of the system when the input is $x[n] = \cos(\frac{\pi}{4}n)$? What is the **output** when the input is $x[n] = 1$ for all n?

Part (e): Computation via Discrete Fourier Transform. Consider an input $x[n]$ which is nonzero for $n = 0, \ldots, N-1$. Set

$$w[n] = \mathrm{DFT}_L^{-1}\Big(H_0^M X\Big)[n] \tag{1}$$

where $H_0 = \mathrm{DFT}_L(h_0)$ and $X = \mathrm{DFT}_L(x)$. For what choices of L is $w[n] = y[n]$?

Part (f): Composition with downsampling. Consider a new system



Consider an input of the form $x[n] = \exp(j\omega_0 n)$. For what choices of ω_0 is the output $y[n] = 0$ for all n?

Answer to Problem 1:

Applying the endpoint relationship for convolution (the convolution of signals supported on $a \leq n \leq b$ and $c \leq n \leq d$ is itself supported on $a+c \leq n \leq b+d$), we begin by noting that $h[n]$ is supported on $0 \leq n \leq 2M$.

Part (a) Yes. The impulse response $h[n]$ is zero for $n < 0$

Part (b) Yes. This is an FIR system, and hence $\|h\|_{\ell^1} < \infty$.

Part (c) We have

$$H(e^{j\omega}) = H_0(e^{j\omega})^M.$$

Moreover,

$$H_0(e^{j\omega}) = -1 + \sqrt{2}e^{-j\omega} - e^{-j2\omega},$$

and so

$$H(e^{j\omega}) = \left(-1 + \sqrt{2}e^{-j\omega} - e^{-j2\omega}\right)^M.$$

Part (d) We start with $x[n] = 1 = e^{j0n}$. We have that

$$y[n] = H(e^{j0})e^{j0n} = (-1 + \sqrt{2} - 1)^M = (\sqrt{2} - 2)^M.$$

For $x[n] = \cos(\frac{\pi}{4}n)$, we compute the system response to $e^{j\frac{\pi}{4}n}$ and $e^{-j\frac{\pi}{4}n}$. Using that $e^{j\frac{\pi}{4}} = \sqrt{2}+j\sqrt{2}$ and $e^{j\frac{\pi}{2}} = j$, we have

$$H_0(e^{j\frac{\pi}{4}}) = -1 + \sqrt{2}(\sqrt{2} + \sqrt{2}j) - j = 0. \tag{2}$$

Similarly,

$$H_0(e^{-j\frac{\pi}{4}}) = -1 + \sqrt{2}(\sqrt{2} - \sqrt{2}j) + j = 0. \tag{3}$$

So the system response to $x[n] = \cos(\frac{\pi}{4}n)$ is $y[n] = 0$.

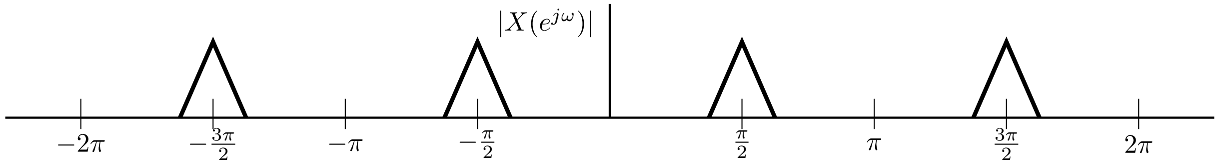
Part (e) We need to set L at least as large as the output length. H has length $2M+1$; X has length N; the output length is $2M+N$, and hence we need to set $L \geq 2M+N$.

Part (f) Let $x_d[n]$ denote the signal after downsampling. Notice that $x_d[n] = x[2n] = \exp(j2\omega_0 n)$. From part (d), for $x_d[n] = \exp(j\nu n)$ the system output is zero whenever $\nu = \pm\pi/4 + 2\pi k$, and hence the overall system output is zero whenever

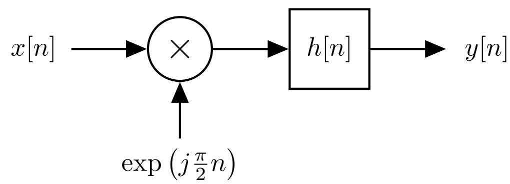
$$\omega_0 = \pm\tfrac{\pi}{8} + \pi k,$$

for some $k \in \mathbb{Z}$.

2. Systems in Frequency Domain. Consider a signal $x[n]$ with $|X(e^{j\omega})|$ shown below:



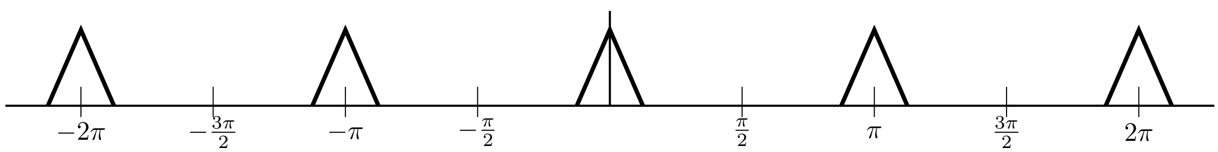
The signal $x[n]$ is input into the following system:



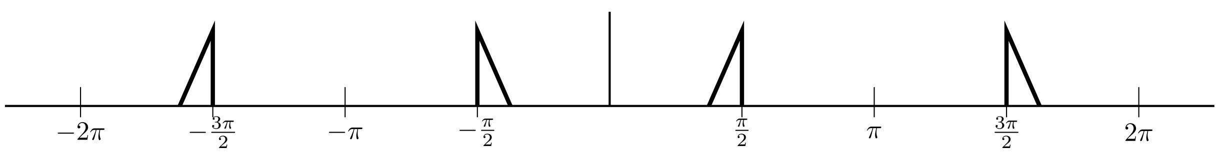
Here, the second block represents an LTI system with *real-valued* impulse response $h[n]$.

For each of the following graphs, indicate whether it could be the magnitude spectrum $|Y(e^{j\omega})|$ of the system output $y[n]$. Please explain why or why not.

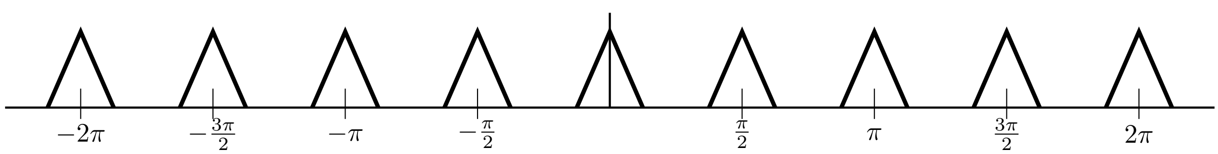
A.



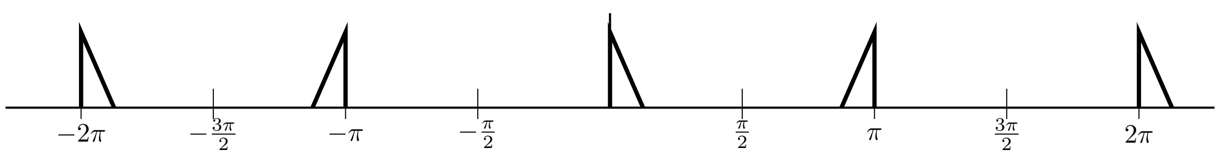
B.



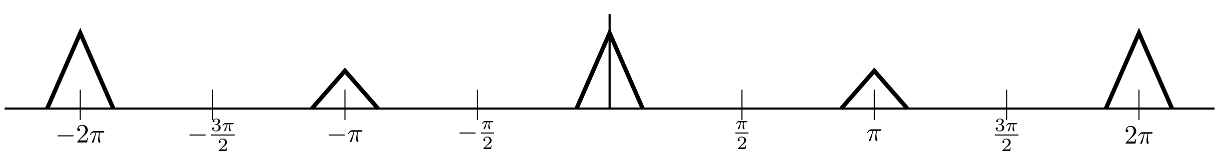
C.



D.



E.



Answer to Problem 2:

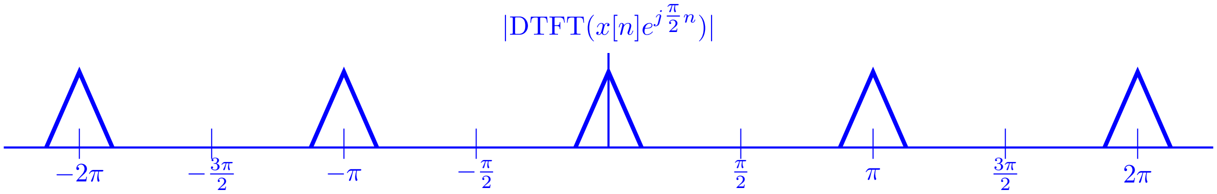
The system applies a two-part processing pipeline, which first modulates by $e^{j\frac{\pi}{2}n}$ and then applies the filter $h[n]$. Using the Fourier relationship

$$e^{j\frac{\pi}{2}n} \mapsto 2\pi \sum_{k=-\infty}^{\infty} \delta\Big(\omega - \tfrac{\pi}{2} - 2\pi k\Big) \tag{4}$$

and the modulation property of the DTFT, we have

$$\begin{aligned} \mathrm{DTFT}\Big\{x[n]e^{j\frac{\pi}{2}n}\Big\}(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\left[2\pi\sum_k \delta\Big(\omega-\theta-\tfrac{\pi}{2}-2\pi k\Big)\right]d\theta \\ &= X(e^{j(\omega-\pi/2)}) \end{aligned}$$

i.e., X is shifted to the right by $\pi/2$ radians, producing the following picture:



In frequency domain, convolution with $h[n]$ is multiplication with $H(e^{j\omega})$. Because h is real-valued, $|H|$ is symmetric. We have

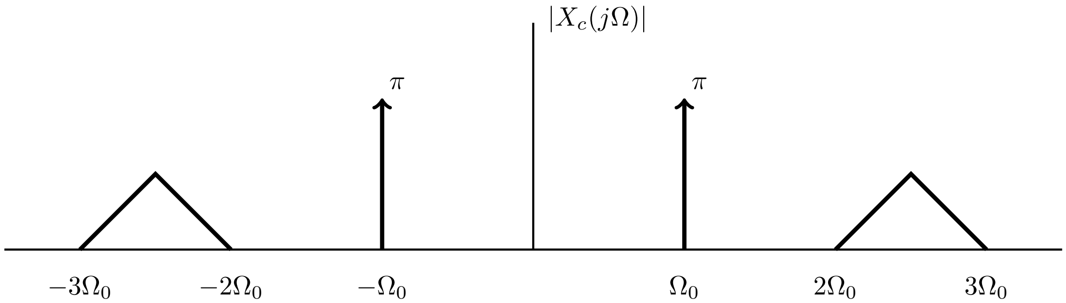
- A. Yes - choose $h[n] = \delta[n]$.
- B. No - graph contains nonzero frequency content around $\pi/2$, $X(e^{j(\omega-\pi/2)})$ is zero in this region.
- C. No - same as B, multiplication by $H(e^{j\omega})$ cannot create nonzero values where $X(e^{j(\omega-\pi/2)})$ is zero.
- D. No - $|X(e^{j(\omega-\pi/2)})|$ is symmetric about $\omega = 0$; $|H(e^{j\omega})|$ is also symmetric and so the product should be symmetric.
- E. Yes - we could choose $h[n]$ such that

$$H(e^{j\omega}) = \begin{cases} 1 & -\pi/2 \le \omega \le \pi/2 \\ 1/2 & \pi/2 \le |\omega| \le \pi \end{cases}$$

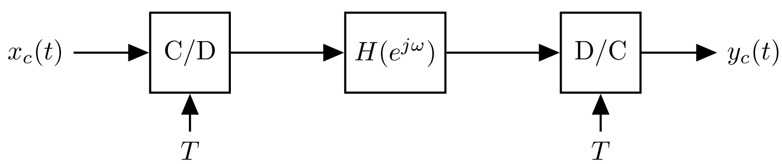
3. Sampling and Filtering. Consider a signal

$$x_c(t) = \cos(\Omega_0 t) + w_c(t),$$

which consists of a "clean" signal $\cos(\Omega_0 t)$ plus a "noise" signal $w_c(t)$ whose Fourier transform $W_c(j\Omega)$ is nonzero only for $2\Omega_0 < |\Omega| < 3\Omega_0$. The Fourier transform of x_c is visualized below:



Consider the following processing pipeline:



Please answer the following questions:

Part (a). For this part, suppose that $h[n] = \delta[n]$. *What is the largest sampling period T for which $y_c(t) = x_c(t)$?*

Note: for Part (a), we want $y_c(t) = x_c(t) = \cos(\Omega_0 t) + w_c(t)$, i.e., $y_c(t)$ should contain *both* the "signal" $\cos(\Omega_0 t)$ and "noise" $w_c(t)$. In Parts (b)-(c), we will remove the noise.

For parts (b)-(c), we attempt to remove the noise, $w_c(t)$, i.e., to make $y_c(t) = \cos(\Omega_0 t)$, by choosing $H(e^{j\omega})$ as an ideal low-pass filter

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < 1.5\Omega_0 T, \\ 0 & 1.5\Omega_0 T \le |\omega| \le \pi. \end{cases} \tag{5}$$

Note that here, the number T in the filter cutoff $1.5\Omega_0 T$ is the *same* as the sampling period T in the C/D and D/C converters.

Part (b). For the T you determined in Part (a), and $H(e^{j\omega})$ specified in (5), is $y_c(t) = \cos(\Omega_0 t)$? Why or why not?

Part (c). With $H(e^{j\omega})$ specified in (5), what is the *largest* sampling period T for which $y_c(t) = \cos(\Omega_0 t)$?

Answer to Problem 3:

Part (a). By Shannon-Nyquist, exact reconstruction occurs when $\Omega_s \geq 6\Omega_0$. Since $T = 2\pi/\Omega_s$, the largest allowable sampling period is $T = \pi/3\Omega_0$.

Part (b). Yes, with this choice, the overall system is equivalent to a low-pass filter

$$H_c(j\Omega) = \begin{cases} 1 & |\Omega| < 1.5\Omega_0 \\ 0 & \text{else} \end{cases} \tag{6}$$

which removes the noise and preserves the signal.

Part (c). The noise will be removed as long as no aliased copy $X_c(j\Omega + k\Omega_s)$ overlaps the interval $(-1.5\Omega_0, 1.5\Omega)$. This is ensured as long as $\Omega_s - 3\Omega_0 \geq 1.5\Omega_0$, i.e., $\Omega_s \geq 4.5\Omega_0$. The largest allowable period is $2\pi/4.5\Omega_0 = \frac{4\pi}{9\Omega_0}$.

Scratch paper: